

# Asymptotic behaviour of the critical value for the contact process with rapid stirring

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## Abstract

We study the behaviour of the contact process with rapid stirring on the lattice  $\mathbb{Z}^d$  in dimensions  $d \geq 3$ . This process was studied earlier by Konno and Katori, who proved results for the speed of convergence of the critical value as the rate of stirring approaches infinity. In this article we improve the results of Konno and Kattori and establish the sharp asymptotics of the critical value in dimensions  $d \geq 3$ .

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# 1 Introduction and the main result

In this paper, we are going to study the behaviour of the so-called contact process with rapid stirring (see [5], [6]). The process is also known as a contact process combined with an exclusion process (see [7, 8]).

DeMasi, Ferrari and Lebowitz (see [2]) studied interacting particle systems on a lattice under the combined influence of spin flip and simple exchange dynamics. They proved that when the stirrings occur on a fast time scale of order  $\varepsilon^{-2}$  the macroscopic density, defined on spatial scale  $\varepsilon^{-1}$ , evolves according to an autonomous nonlinear diffusion-reaction equation. Using the connection between a convergent sequence of such particle systems to a solution of a reaction-diffusion equation, found by DeMasi, Ferrari and Lebowitz, Durrett and Neuhauser (in [3]) proved results about the existence of phase transitions when the stirring rate is large that apply to many different systems.

The contact process with rapid stirring was studied by Konno (see [6]), who described it via a system of interacting particles, on a lattice  $\mathbb{Z}^d$  (while for the proofs the process on the rescaled lattice  $\mathbb{Z}^d/\sqrt{N}$  was considered). The state of the process at time  $t$  is given by a function  $\eta_t^N : \mathbb{Z}^d \rightarrow \{0, 1\}$ , where the value of  $\eta_t^N(x)$  is determined by the number of particles present at  $x$  at time  $t$ . In this setting particles die at rate 1, and give birth, onto the closest neighbouring sites, at rate  $\lambda$ . In addition, values of  $\eta^N$  at two neighbouring sites are exchanged at rate  $N$  (stirring rate), and all the above mechanisms are independent. The primary goal of [6] was to improve the result of Durrett and Neuhauser, who showed the following:

**Theorem 1.1 (Durrett, Neuhauser, [3])** *Let  $\{\eta_t^N\}_{t \geq 0}$  be the set-valued contact process with stirring, with the dynamics as described above, starting with a single occupied site at the origin. Let  $\Omega_\infty = \{\eta_t^N \neq \emptyset, \forall t \geq 0\}$ ,  $\rho_\lambda^N = \mathbf{P}[\Omega_\infty]$ , and let*

$$\lambda_c(N) = \inf\{\lambda \geq 0 : \rho_\lambda^N > 0\}.$$

As  $N \rightarrow \infty$ ,

- a.  $\lambda_c(N) \rightarrow 1$ .
- b. If  $\lambda > 1$ , then  $\rho_\lambda^N \rightarrow (\lambda - 1)/\lambda$ .

Konno used the methods of [1] to get a more detailed picture of the critical value  $\lambda_c(N)$  as the stirring rate  $N$  approaches infinity. The main result of [6] can be stated as follows.

**Theorem 1.2 (Konno, [6])** *For all  $x > 0$ , let*

$$\varphi_d(x) = \begin{cases} 1/x^{1/3}, & d = 1, \\ \log x/x, & d = 2, \\ 1/x, & d \geq 3. \end{cases}$$

*Then we have*

$$\lambda_c(N) - 1 \approx C_\star \varphi_d(N)$$

*where  $\approx$  means that if  $C_\star$  is small (large) then the right-hand side of the above is a lower (upper) bound of the left-hand side for large enough  $N$ .*

Moreover, Theorem 1.2 was refined in dimensions  $d \geq 3$ : with the help of result of Katori (see [5]), it was shown in [6] that

$$\begin{aligned} \frac{1}{(2d)(2d-1)} &\leq \liminf_{N \rightarrow \infty} N (\lambda_c(N) - 1) \\ &\leq \limsup_{N \rightarrow \infty} N (\lambda_c(N) - 1) \leq \frac{G(0,0) - 1}{2d}, \end{aligned} \tag{1.1}$$

where  $G(\cdot, \cdot)$  is the Green's function for the simple random walk on  $\mathbb{Z}^d$ .

The main goal of this article is to show that, in fact, the lower bound in (1.1) can be improved to the value  $\frac{G(0,0)-1}{2d}$ . By this, we get the sharp asymptotics of the critical value  $\lambda_c(N)$ . Before we state our main result, we need another piece of notation. Let  $\partial_d(x) = \{y \in \mathbb{Z}^d : \|y - x\|_1 = 1\}$  denote "the neighbourhood of  $x$  in  $\mathbb{Z}^d$ , excluding  $x$ ". Then  $\partial_d \equiv \partial_d(0)$  will denote the neighbourhood of the origin.

The main result of this article is

**Theorem 1.3** *Let  $d \geq 3$ . Then*

$$\lambda_c(N) - 1 \sim \frac{\vartheta}{N}, \text{ as } N \rightarrow \infty,$$

where

$$\vartheta = \frac{1}{4d^2} \sum_{n=1}^{\infty} \mathbf{P}[V_n \in \partial_d], \quad (1.2)$$

$\{V_n\}_{n \geq 0}$  is a symmetric random walk on  $\mathbb{Z}^d$  starting at the origin, and  $\sim$  means the ratio approaches 1, as  $N$  approaches  $\infty$ .

To connect the result in the theorem with (1.1) let us state a simple

**Lemma 1.4**

$$2d\vartheta = G(0, 0) - 1.$$

**Remark 1.5** *The lemma implies that our sharp asymptotics for the critical value coincides with the upper bound for the critical value in (1.1).*

**Proof of Lemma 1.4** By the Markov property

$$\begin{aligned} G(0, 0) &= 1 + \frac{1}{2d} \sum_{n=1}^{\infty} \mathbf{P}[V_{n-1} \in \partial_d] \\ &= 1 + \frac{1}{2d} \sum_{n=1}^{\infty} \mathbf{P}[V_n \in \partial_d], \end{aligned}$$

where the last line follows since  $V_0 = \mathbf{0}$ . □

Let us say a few words about the proofs. The structure of Konno's proofs of Theorem 1.2 follows the ideas of Bramson et alii (see [1]), who studied the long range contact process (LRCP) in a limiting régime, when the range  $M$  of the contact process, goes to infinity. The set up of Konno is very similar to that of Bramson et alii in [1], with the difference that the stirring speed goes to infinity, and not the range. Note that in [1], the authors were able to prove an asymptotic result for LRCP which was later improved by Durrett and Perkins (see [4]) where a sharp asymptotics for the convergence of the critical value was obtained for dimensions  $d \geq 2$ . To prove the sharp asymptotics, Durrett and Perkins, in [4], had further rescaled space and time and proved weak convergence of the rescaled processes to super-Brownian motion with drift. This convergence almost immediately gives the lower bound for the critical value for LRCP. As for the upper bound, Durrett and Perkins bounded the LRCP from below by an oriented percolation process (for this they used again convergence to super-Brownian motion), and, by this, the

upper bound for the critical value was derived.

As for the proof of our main result—Theorem 1.3—it follows immediately that from (1.1) and Lemma 1.4, that it is sufficient to prove just the lower bound for the critical value. This makes the proofs far less complicated than those in [4]. In fact, we prove our result without proving a weak convergence result of rescaled processes to super-Brownian motion, which was one of the main technical ingredients of the proofs in [4].

The rest of the paper is organized as follows. Formal definitions of the contact process and "speeded-up" contact process are given in Section 2. Theorem 1.3 is proved in Section 3.

## 2 Formal definitions

Before we proceed to the proofs, let us give formal definitions in this section.

The contact process with rapid stirring takes place on the lattice  $\mathbb{Z}^d$ . Fix parameter  $\theta$  for this process. The state of the process at time  $t$  is given by a function  $\eta_t^N : \mathbb{Z}^d \rightarrow \{0, 1\}$ , where the value of  $\eta_t^N(x)$  is determined by the number of particles present at  $x$  at time  $t$ . Assume that  $\eta_0^N = \delta_0$ . Independently of each other:

1. particles die at rate 1 without producing offspring;
2. particles split into two at rate  $1 + \theta/N$ . If split occurs at  $x \in \mathbb{Z}^d$ , then one of the particles replaces the parent, while the other is sent to a site  $y$  chosen according to a uniform distribution on  $\tilde{\partial}_d(x)$  (the nearest neighbouring sites of  $x$ ). If a newborn particle lands on an occupied site, its birth is suppressed;
3. for each  $x, y \in \mathbb{Z}^d$ , with  $x - y \in \tilde{\partial}_d$ , the values of  $\eta^N$  at  $x$  and  $y$  are exchanged at rate  $N$  (stirring).

Just to clarify, when we say that events occur at a certain rate, we mean that times between events are independent exponential random variables with that rate. Let us also make a comment about rule 3 above. In terms of particles dynamics, it means that whenever exchange between sites  $x$  and  $y$  occurs, a particle at  $x$  (if exists) jumps to  $y$ , and at the same time a particle

at  $y$  (if exists) jumps to  $x$ . If one follows the motion of a typical particle, then, in the absence of branch events, it undergoes a symmetric random walk on  $\mathbb{Z}^d$ , with jumps at rate  $2dN$ .

For our proofs it will be convenient to deal with the speeded-up contact process  $\hat{\eta}_t^N : \mathbb{Z} \rightarrow \mathbb{N} \cup \{0\}$ . This process is defined as  $\hat{\eta}_t^N = \eta_{Nt}^N, t \geq 0$ . Clearly,  $\hat{\eta}_0^N = \delta_0$ . Obviously this process obeys the same rules as  $\eta^N$ , just all the events occur with rate multiplied by  $N$ . In particular, particles die and split (if possible) with rates  $N$  and  $N + \theta$  respectively; stirring between any two neighbouring sites occurs with rate  $N^2$ .

### 3 Proof of Theorem 1.3

As we have mentioned already, (1.1) and Lemma 1.4 imply that Theorem 1.3 will follow from the lower bound for  $\lambda_c(N) - 1$  obtained in the next proposition.

**Proposition 3.1** *For  $d \geq 3$*

$$\liminf_{N \rightarrow \infty} \frac{\lambda_c(N) - 1}{\vartheta/N} \geq 1.$$

In fact, Proposition 3.1 follows easily from the following crucial result. Recall that  $\hat{\eta}_t^N = \eta_{Nt}^N, t \geq 0$ , was defined in Section 2.

**Proposition 3.2** *Fix an arbitrary  $\theta < \vartheta$ . For any  $t_0 > 0$  there exist  $N_\theta > 0$ , such that, for all  $N > N_\theta$  and  $t > t_0$ ,*

$$\hat{m}_t^N \equiv \mathbf{E} |\hat{\eta}_t^N| \leq e^{\frac{1}{2}(\theta - \vartheta)t},$$

*where  $|\cdot|$  denote the total number of particles in the process.*

Now we are ready to finish the

#### Proof of Proposition 3.1

Fix  $\theta < \vartheta$ , and choose  $N_\theta$  as in Proposition 3.2. Then, by this proposition, and the fact that the number of particles is an integer, we have that

$$\begin{aligned}
\mathbf{P} [|\hat{\eta}_t^N| = 0] &\geq 1 - \hat{m}_t^N \\
&= 1 - e^{\frac{1}{2}(\vartheta - \theta)} \\
&\rightarrow 1,
\end{aligned}$$

as  $t \rightarrow \infty$ , for all  $N > N_\theta$ .

From this, it follows immediately that for  $N > N_\theta$ ,  $\{\hat{\eta}_t^N\}_{t \geq 0}$  dies out in finite time, with probability one. The same happens with  $\{\eta_t^N\}_{t \geq 0}$  with probability one.

Thus we have shown that for any  $\theta < \vartheta$ , there exists an  $N_\theta$  such that for every  $N > N_\theta$ ,

$$\mathbf{P} [\eta_t^N \neq 0, \text{ for all } t > 0 \mid \eta_0^N = \delta_0] = 0.$$

Therefore,

$$\inf\{\theta : \mathbf{P} [\eta_t^N \neq 0 \text{ for all } t > 0 \mid \eta_0^N = \delta_0] > 0\} \geq \vartheta,$$

and, by definition of  $\lambda_c(N)$ , the proof of Proposition 3.1 is finished.  $\square$

**Remark 3.3** *By Lemma 1.4 and (1.1), we have proven Theorem 1.3 modulo Proposition 3.2.*

The rest of this section is devoted to the proof of Proposition 3.2. Before we proceed to its proof, let us first define some additional notation.

- Denote particles by Greek letters  $\alpha, \beta, \gamma$  with the convention that  $\alpha_0$  is the ancestor of  $\alpha$  in generation 0.
- Consider that particles change names every time an event they branch (die or split).
- Let  $\alpha \wedge \beta$  denote the most recent common ancestor of  $\beta$  and  $\gamma$ .
- In the “speeded-up” process  $\{\hat{\eta}_t^N\}_{t \geq 0}$ , let  $T_\alpha$  denote that time at which  $\alpha$  branches (dies or splits) and let  $B_t^\alpha$  be the location of  $\alpha$  lineage at  $t$  with the convention that  $B_t^\alpha = \Delta$  if the particle is not alive at  $t$ .

- Let  $\zeta_\alpha$  be the indicator of the event that a split occurred at  $T_\alpha$  but one of  $\alpha$ 's children imminently collided with another particle and died "at birth".

The idea behind the proof is that in order to bound from above the total mass of the process we can ignore collisions between distant relatives. To this end, we define a sequence of times  $\tau_N$  such that collisions between relatives farther related than  $\tau_N$ , in the process  $\hat{\eta}_t^N$ , can be ignored.

Let

$$\tau_N = \frac{\ln N}{N^2} > 0. \quad (3.1)$$

Let

$$Z_1(t) = 1(T_1 \in [t, t + \tau_N), B_{t+\tau_N}^\beta - B_{t+\tau_N}^\gamma \in \bar{\partial}_d),$$

where  $\beta$  and  $\gamma$  are the children of  $1 \equiv \beta_0 = \gamma_0$  – the first particle.  $Z_1(t)$  is the indicator of the event that the lineage of 1 has exactly one splitting event in  $[t, t + \tau_N)$ , no deaths and its two offsprings ( $\beta$  and  $\gamma$ ) are alive and neighbours at time  $t + \tau_N$ . Set  $Z_1 \equiv Z_1(0)$ .

Before we proceed to the actual proof of Proposition 3.2 we will need an auxiliary result. Let  $\{V_t^N\}_{t \geq 0}$  be a continuous time, symmetric random walk on  $\mathbb{Z}^d$  jumping with rate  $4dN^2$  and starting at the origin. Let  $\{W_t^N\}_{t \geq 0}$  be a continuous time Markov chain taking values in  $\mathbb{Z}^d$ , starting from the origin, and evolving as follows. If  $W_t^N = x \in \bar{\partial}_d$  then, with rate  $(4d - 1)N^2$ ,  $W^N$  makes a jump to  $y \in \mathbb{Z}^d$ , whereas with probability  $\frac{4d-2}{4d-1}$   $y$  is chosen uniformly from  $\bar{\partial}_d(x) \setminus \{\mathbf{0}\}$ , and with probability  $\frac{1}{4d-1}$ ,  $y = -x$ . If  $W_t^N = x \notin \bar{\partial}_d$ , then, with rate  $4dN^2$ ,  $W^N$  makes a jump to  $y$  uniformly distributed in  $\bar{\partial}_d(x)$ .

Note that  $W_t^N$  describes the behaviour of the difference in locations of two typical particles in the process  $\hat{\eta}^N$  in the absence of branching events. Such particles move around independently like symmetric random walks, with jumps rates  $2dN^2$ , until they become neighbours. While they are neighbours, their behaviour is dictated by the stirring rules.

**Lemma 3.4**

$$\mathbf{E} \left[ \int_0^t V_s^N 1(V_s^N \in \bar{\partial}_d) ds \right] = \mathbf{E} \left[ \int_0^t W_s^N 1(W_s^N \in \bar{\partial}_d) ds \right].$$



**Proof** Given  $V_t^N \in \mathfrak{D}_d$ , define  $T_v$  to be the time  $V^N$  spends in  $\mathfrak{D}_d$  before leaving the set  $\mathfrak{D}_d \cup \{\mathbf{0}\}$ . Then clearly,

$$T_v = \sum_{i=1}^R \epsilon_i,$$

where  $\epsilon_i$ s are independent random variables distributed according to exponential distribution with rate  $4dN^2$  and  $R$  is independent of them and is geometric with parameter  $\frac{2d-1}{2d}$ . Clearly  $T_v$  is exponentially distributed with  $\mathbf{E}T_v = (2(2d-1)N^2)^{-1}$ .

Similarly, at every visit to  $\mathfrak{D}_d$ ,  $W_t^N$  spends in  $\mathfrak{D}_d$  a time  $T_w = \sum_{i=1}^{R'} \epsilon'_i$ , where  $R'$  is geometric with parameter  $\frac{4d-2}{4d-1}$  and  $\epsilon'_i$  are independent exponentially distributed random variables with rate  $(4d-1)N^2$  and independent of  $R'$ . Thus,  $T_w$  is exponentially distributed with the mean

$$\left( (4d-1)N^2 \frac{4d-2}{4d-1} \right)^{-1} = (2(2d-1)N^2)^{-1}.$$

Thus we may couple the processes together by setting them to be equal, every time they exit  $\mathfrak{D}_d \cup \{\mathbf{0}\}$ , and as this does not change time they spend in  $\mathfrak{D}_d$  the result follows.  $\square$

### Lemma 3.5

$$\lim_{N \rightarrow \infty} N\mathbf{E}[Z_1] = d\vartheta.$$

**Proof** First we calculate the probability of  $F_1$ , the event that there is exactly one birth in 1's lineage in  $\tau_N$  units of time and no deaths on any of the branches. As the births in  $\hat{\eta}^N$  process occur according to a Poisson process with rate  $N + \theta$ , we have that

$$\begin{aligned} \mathbf{P}[F_1] &= (N + \theta)\tau_N e^{-(N+\theta)\tau_N} e^{-N\tau_N} \cdot \frac{1}{\tau_N} \int_0^{\tau_N} e^{-(2N+\theta)s} ds \quad (3.2) \\ &= \frac{N + \theta}{2N + \theta} e^{-(2N+\theta)\tau_N} (1 - e^{-(2N+\theta)\tau_N}). \end{aligned}$$

$$\mathbf{E}[Z_1] = \mathbf{P}[B_{\tau_N}^\beta - B_{\tau_N}^\gamma \in \mathfrak{D}_d | F_1] \mathbf{P}[F_1],$$

where  $\beta$  and  $\gamma$  are the offspring of 1 alive at  $\tau_N$ .

$$\mathbf{P} [B_{\tau_N}^\beta - B_{\tau_N}^\gamma \in \mathfrak{d}_d | F_1] = \frac{1}{\tau_N} \int_0^{\tau_N} \mathbf{P} [W + W_{\tau_N-t}^N \in \mathfrak{d}_d] dt, \quad (3.3)$$

where  $W$  is uniform on  $\mathfrak{d}_d$  and is the difference of positions of the two children of 1, right after the split;  $\{W_t^N\}_{t \geq 0}$  is a continuous time Markov process defined in Lemma 3.4 independent of  $W$ .

Change the variable in the integral, use Lemma 3.4 to get that (3.3) is equal to

$$\begin{aligned} & \frac{1}{\tau_N} \int_0^{\tau_N} \mathbf{P} [W + V_s^N \in \mathfrak{d}_d] ds \\ &= \frac{1}{\tau_N} \int_0^{\tau_N} \sum_{n=0}^{\infty} \mathbf{P} [W + V_n \in \mathfrak{d}_d] \frac{e^{-4dN^2s} (4dN^2s)^n}{n!} ds, \end{aligned} \quad (3.4)$$

where  $V_n$  is a simple symmetric random walk on  $\mathbb{Z}^d$  independent of  $W$ .

Now let  $\pi(u)$  be a Poisson random variable with mean  $u$ , and define

$$h(u) \equiv \mathbf{P} [W + V_{\pi(u)} \in \mathfrak{d}_d].$$

Thus (3.4) can be written as

$$\frac{1}{\tau_N} \int_0^{\tau_N} h(4dN^2s) ds. \quad (3.5)$$

So, from (3.2), (3.5), and (3.1) we have that

$$\begin{aligned} \mathbf{E} Z_1 &= \frac{N + \theta}{2N + \theta} e^{-2(N+\theta)\tau_N} (1 - e^{-(2N+\theta)\tau_N}) \frac{1}{\tau_N} \int_0^{\tau_N} h(4dN^2s) ds \\ &= \frac{1}{4dN^2\tau_N} \frac{N + \theta}{2N + \theta} e^{-(2N+\theta)\tau_N} (1 - e^{-(2N+\theta)\tau_N}) \int_0^{4dN^2\tau_N} h(r) dr, \end{aligned} \quad (3.6)$$

where the last equality follows from changing the variable inside the integral.

Now let  $N \rightarrow \infty$ , use the Taylor expansion for the exponential and the monotone convergence theorem to get that

$$\lim_{N \rightarrow \infty} N \mathbf{E} Z_1 = \frac{1}{4d} \int_0^\infty \mathbf{P} [W + V_{\pi(s)} \in \mathfrak{D}_d] ds.$$

The times between jumps of  $V_{\pi(s)}$  are exponential with mean 1. Therefore

$$\lim_{N \rightarrow \infty} N \mathbf{E} Z_1(\tau_N) = \frac{1}{4d} \sum_{n=1}^\infty \mathbf{P} [V_n \in \mathfrak{D}_d]. \quad (3.7)$$

This finishes the proof of this lemma.  $\square$

### Proof of Proposition 3.2

Set  $m_t^N = \mathbf{E} |\eta_t^N|$ . From (1.5) of [6], we have

$$\begin{aligned} m_{Nt}^N &= 1 + \int_0^{Nt} \frac{\theta}{N} m_s^N ds - \frac{1}{2d} \int_0^{Nt} I_s^N ds \\ &= 1 + \int_0^t \theta m_{Ns}^N ds - \frac{1}{2d} \int_0^t N I_{Ns}^N ds, \end{aligned}$$

where  $I_s^N$  is twice the expected number of pairs of neighbours in  $\eta_t^N$  at time  $s$ . Let  $\mathcal{A}(s)$  be the set of particles which are alive at  $s$  in  $\hat{\eta}_t^N$ , and

$$\hat{I}_s^N = \mathbf{E} \left[ \sum_{\alpha, \beta \in \mathcal{A}(s)} 1(B_s^\alpha - B_s^\beta \in \mathfrak{D}_d) \right]$$

be twice the expected number of pairs of neighbours in  $\hat{\eta}_t^N$  at time  $s$ .

Thus, with  $\hat{m}_t^N = m_{Nt}^N = \mathbf{E} |\eta_{Nt}^N|$ , we have

$$\hat{m}_t^N = 1 + \int_0^t \theta \hat{m}_s^N ds - \frac{1}{2d} \int_0^t N \hat{I}_s^N ds. \quad (3.8)$$

Similarly to the definition of  $Z_1$ , let

$$Z_\alpha(t) = 1(T_\alpha \in [t, t + \tau_N), B_{t+\tau_N}^\beta - B_{t+\tau_N}^\gamma \in \mathfrak{D}_d),$$

where  $\beta$  and  $\gamma$  are the children of  $\alpha$ . Further, let  $\zeta_\alpha(t)$  be the indicator of the event that one of  $\alpha$ 's children created in  $[t, t + \tau_N)$  died as a result of a

collision at the time of its birth  $T_\alpha$ . For any  $s \geq \tau_N$ , let us now give a lower bound for

$$\mathbf{E}[Z_\alpha(s - \tau_N) | \mathcal{F}_{s-\tau_N}].$$

Note that given  $\mathcal{F}_{s-\tau_N}$ , for  $s \geq \tau_N$  and  $\alpha \in \mathcal{A}(s - \tau_N)$ ,  $Z_\alpha(s - \tau_N)$  is stochastically less than  $Z_1$ , since one of  $\alpha$ 's children created in  $[t, t + \tau_N)$  may be killed as a result of a collision at the time of its birth  $T_\alpha$ . However, if we “return” the killed children back, then we easily get that, conditionally on  $\mathcal{F}_{s-\tau_N}$ , for  $s \geq \tau_N$  and  $\alpha \in \mathcal{A}(s - \tau_N)$ ,  $Z_\alpha(s - \tau_N) + \zeta_\alpha(s - \tau_N)$  is stochastically greater than  $Z_1$ . Therefore we immediately get, that for  $s \geq \tau_N$ ,  $\alpha \in \mathcal{A}(s - \tau_N)$ ,

$$\mathbf{E}[Z_\alpha(s - \tau_N) | \mathcal{F}_{s-\tau_N}] \geq \mathbf{E}[Z_1] - \mathbf{E}[\zeta_\alpha(s - \tau_N) | \mathcal{F}_{s-\tau_N}].$$

Use this to get, for  $s \geq \tau_N$ ,

$$\begin{aligned} \hat{I}_s^N &\geq \mathbf{E} \left[ \sum_{\substack{\alpha, \beta \in \mathcal{A}(s) \\ T_{\alpha \wedge \beta} > s - \tau_N}} 1(B_s^\alpha - B_s^\beta \in \mathfrak{D}_d) \right] \\ &= 2\mathbf{E} \left[ \sum_{\alpha \in \mathcal{A}(s - \tau_N)} \mathbf{E}[Z_\alpha(s - \tau_N) | \mathcal{F}_{s-\tau_N}] \right] \\ &\geq 2\mathbf{E} \left[ \sum_{\alpha \in \mathcal{A}(s - \tau_N)} (\mathbf{E}[Z_1] - \mathbf{E}[\zeta_\alpha(s - \tau_N) | \mathcal{F}_{s-\tau_N}]) \right] \\ &= 2\mathbf{E}[\mathcal{A}(s - \tau_N) | \mathbf{E}[Z_1]] - 2\mathbf{E} \left[ \sum_{\alpha \in \mathcal{A}(s - \tau_N)} \zeta_\alpha(s - \tau_N) \right]. \end{aligned} \tag{3.9}$$

Note now that for  $s < \tau_N$  we have the same result through a similar calculation (with  $\mathcal{A}(0)$  instead of  $\mathcal{A}(s - \tau_N)$ ) and a slight modification of  $Z_1$ .

Using Lemma 3.5 and (3.9), (3.8) gives

$$\hat{m}_t^N \leq 1 + \theta \int_0^t \hat{m}_s^N ds - \left( \vartheta - \frac{\varepsilon}{4} \right) \int_0^t \hat{m}_{(s-\tau_N)_+}^N ds + \frac{1}{d} \int_0^t \Xi_s^N ds, \tag{3.10}$$

where  $\Xi_s^N = N\mathbf{E} \sum_{\alpha \in \mathcal{A}((s-\tau_N)_+)} \zeta_\alpha((s - \tau_N)_+)$ .

On the other hand, let  $\tau_N^s = s - (s - \tau_N)_+$ , then

$$\begin{aligned}
\Xi_s^N &\leq N \mathbf{E} \left[ \sum_{\alpha \in \mathcal{A}((s-\tau_N)_+)} \mathbf{P} \left( T_\alpha \in [(s-\tau_N)_+, s] \middle| \mathcal{F}_{(s-\tau_N)_+} \right) \right. \\
&\quad \times \frac{1}{2d\tau_N^s} \mathbf{E} \left[ \int_{(s-\tau_N)_+}^s \sum_{\gamma \in \mathcal{A}(r)} 1(B_r^\alpha - B_r^\gamma \in \mathfrak{D}_d) dr \middle| \mathcal{F}_{(s-\tau_N)_+} \right] \left. \right] \\
&= \frac{N}{2d\tau_N^s} (2N + \theta) \tau_N^s e^{-(2N+\theta)\tau_N^s} \mathbf{E} \left[ \int_{(s-\tau_N)_+}^s \sum_{\alpha, \gamma \in \mathcal{A}(r)} 1(B_r^\alpha - B_r^\gamma \in \mathfrak{D}_d) dr \right] \\
&\leq CN^2 \int_{(s-\tau_N)_+}^s \hat{I}_r^N dr.
\end{aligned} \tag{3.11}$$

Changing the order of integration yields

$$\begin{aligned}
\int_0^t \int_{(s-\tau_N)_+}^s \hat{I}_r^N dr ds &= \int_0^t \int_r^{(r+\tau_N) \wedge t} \hat{I}_s^N ds dr \\
&\leq \tau_N \int_0^t \hat{I}_r^N dr \\
&\leq 2dN^{-1}\tau_N \left( 1 + \int_0^t \theta \hat{m}_s^N ds \right),
\end{aligned} \tag{3.12}$$

where the last inequality follows from (3.8).

From (3.11) and (3.12) we get that

$$\frac{1}{d} \int_0^t \Xi_s^N ds \leq 2N^{-1}\tau_N + 2\theta N^{-1} \int_0^t \hat{m}_s^N ds.$$

Thus, (3.10) can be rewritten as

$$\hat{m}_t^N \leq 1 + 2N^{-1}\tau_N + \theta (1 + 2N^{-1}) \int_0^t \hat{m}_s^N ds - \left( \vartheta - \frac{\varepsilon}{4} \right) \int_0^t \hat{m}_{(s-\tau_N)_+}^N ds. \tag{3.13}$$

Clearly, from (3.8),

$$\hat{m}_t^N \leq 1 + \theta \int_0^t \hat{m}_s^N ds, \quad \forall t \geq 0.$$

Therefore, as 1 is a non-decreasing function, Gronwall's lemma gives that

$$\hat{m}_t^N \leq e^{\theta(t-r)} \hat{m}_r^N, \quad \forall r \in [0, t].$$

Thus,

$$\hat{m}_{(t-\tau_N)_+}^N \geq \hat{m}_t^N e^{-\theta\tau_N}, \quad \forall t \geq 0.$$

By choosing  $N_1$  large enough so that for any  $N \geq N_1$  such that  $(\vartheta - \varepsilon/4)e^{-\theta\tau_N} \geq (\vartheta - \varepsilon/2)$  we get,

$$\begin{aligned} \hat{m}_t^N &\leq 1 + 2N^{-1}\tau_N + \theta(1 + 2N^{-1}) \int_0^t \hat{m}_s^N ds - (\vartheta - \varepsilon/2) \int_0^t \hat{m}_s^N ds \\ &= 1 + 2N^{-1}\tau_N + (\theta - \vartheta + \varepsilon/2 + 2\theta N^{-1}) \int_0^t \hat{m}_s^N ds. \end{aligned}$$

Again by Gronwall's lemma

$$\hat{m}_t^N \leq (1 + 2N^{-1}\tau_N) e^{(\theta - \vartheta + \varepsilon/2 + 2\theta N^{-1})t}, \quad \forall t > 0, N \geq N_1.$$

Since  $\theta < \vartheta$  we can choose  $\varepsilon = (\vartheta - \theta)/2$  and  $N_2 > N_1 \vee \frac{16\theta}{\vartheta - \theta}$  to get

$$\hat{m}_t^N \leq (1 + 2N^{-1}\tau_N) e^{\frac{5}{8}(\theta - \vartheta)t}, \quad \forall N \geq N_2, t > 0.$$

Then choose  $t_0 \geq 0$  and  $N_\theta \geq N_2$ , such that

$$(1 + 2N_\theta^{-1}\tau_N) e^{-\frac{1}{8}(\vartheta - \theta)t_0} \leq 1,$$

and hence

$$\hat{m}_t^N \leq e^{\frac{1}{2}(\theta - \vartheta)t}, \quad \forall N \geq N_\theta, t > t_0.$$

□

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